

EMBEDDING ENTROPIC ALGEBRAS INTO SEMIMODULES AND MODULES

M. M. STRONKOWSKI

ABSTRACT. An algebra is entropic if its basic operations are homomorphisms. The paper is focused on representations of such algebras. We prove the following theorem: An entropic algebra without constant basic operations which satisfies so called Szendrei identities and such that all its basic operations of arity at least two are surjective is a subreduct of a semimodule over a commutative semiring. Our theorem is a straightforward generalization of Ježek's and Kepka's theorem for groupoids. As a consequence we obtain that a mode (entropic and idempotent algebra) is a subreduct of a semimodule over a commutative semiring if and only if it satisfies Szendrei identities. This provides a complete solution to the problem in mode theory asking for a characterization of modes which are subreducts of semimodules over commutative semirings. In the second part of the paper we use our theorem to show that each entropic cancellative algebra is a subreduct of a module over a commutative ring. It extends a theorem of Romanowska and Smith about modes.

1. INTRODUCTION

An algebra (A, Ω) is *entropic* if each basic operation $\omega \in \Omega$ is a homomorphism from (A^n, Ω) into (A, Ω) , where n is the arity of ω . This is equivalent to the satisfaction of all *entropic identities*

$$\begin{aligned} (\varepsilon_{\mu, \omega}) \quad & \mu(\omega(x_1^1, \dots, x_n^1), \dots, \omega(x_1^m, \dots, x_n^m)) \\ & \approx \omega(\mu(x_1^1, \dots, x_n^1), \dots, \mu(x_n^1, \dots, x_n^m)), \end{aligned}$$

where $\mu \in \Omega$ and m is its arity. (Semi)modules over commutative (semi)rings are entropic algebras. The main aim of this article is to show that many entropic algebras are in fact quite close to these examples. We use the symbol \mathbf{E} to denote the variety of entropic algebras (of a fixed similarity type).

An efficient way of describing the structure of an algebra is to embed it into another one, usually with a better known and richer structure. The paper contains two results of this sort.

Firstly, we prove that if an entropic algebra has no constant basic operations, its non-unary basic operations are onto and if it satisfies Szendrei identities, then it is a subreduct of a semimodule over a commutative semiring (Theorem 11).

Let us clarify the notions which appear in the previous sentence. An algebra (A, Ω) is a *reduct* of an algebra (A, Φ) if each operation from Ω is a term operation of the algebra (A, Φ) . Therefore taking a reduct is a partial forgetting of the

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algebraic structure. A *subreduct* is a subalgebra of a reduct. Thus our task is to enrich the algebraic structure of considered algebras.

For a basic operation $\omega \in \Omega$ and natural numbers $1 \leq i, j \leq n$, where n is the arity of ω , consider the identity

$$(\sigma_{\omega}^{i,j}) \quad \omega(\omega(x_1^1, \dots, x_n^1), \dots, \omega(x_1^n, \dots, x_n^n)) \\ \approx \omega(\omega(\pi_j^i(x_1^1), \dots, \pi_j^i(x_n^1)), \dots, \omega(\pi_j^i(x_1^n), \dots, \pi_j^i(x_n^n))),$$

where π_j^i is a permutation of variables which transposes x_j^i with x_i^j and leaves other variables fixed. As an example $\sigma_{\omega}^{1,2}$, where ω is ternary, is given by

$$\omega(\omega(x_1^1, \boxed{x_2^1}, x_3^1), \omega(\boxed{x_1^2}, x_2^2, x_3^2), \omega(x_1^3, x_2^3, x_3^3)) \\ \approx \omega(\omega(x_1^1, \boxed{x_1^2}, x_3^1), \omega(\boxed{x_2^1}, x_2^2, x_3^2), \omega(x_1^3, x_2^3, x_3^3)).$$

Here only framed variables change their positions. Note that if the arity of ω is two, then the identities $\sigma_{\omega}^{1,2}$ and $\varepsilon_{\omega, \omega}$ coincide. All $\sigma_{\omega}^{i,j}$ are called *Szendrei identities* (they appeared in the paper [35] by Á. Szendrei) and algebras satisfying them are called *Szendrei algebras*. We use the symbol **SZE** to denote the variety of Szendrei entropic algebras.

Theorem 11 is connected to the following problem posed by A. Romanowska (see [24, p. 543] and [21, Problem 8.11]):

Problem 1. Is each mode a subreduct of a semimodule over a commutative semiring?

A *mode* is an algebra (A, Ω) without constant basic operations which is entropic and *idempotent*, meaning that it satisfies all *idempotent identities*

$$(\iota_{\omega}) \quad \omega(x, \dots, x) \approx x,$$

where $\omega \in \Omega$. While entropic algebras are characterized by the fact that all their term operations are homomorphisms, modes are characterized by the property that all their polynomial operations are homomorphisms.

It is straightforward to verify that each subreduct of a semimodule over a commutative semiring satisfies Szendrei identities. Thus Problem 1 was split by N. Dojer into two:

Problem 2 (Problem 1 in [4]). Is every mode a Szendrei mode?

Problem 3 (Problem 2 in [4]). Is every Szendrei mode a subreduct of a semimodule over a commutative semiring?

Problem 2 was solved negatively in [31], where it was shown that a free mode of rank two with a basic operation of arity at least three does not satisfy Szendrei identities. Later [14] a family of non-Szendrei modes was found among differential modes. In fact, it appeared that Szendrei differential modes form quite a narrow class comparing to the variety of differential modes [18]. Theorem 11 gives a positive solution to Problem 3 (Corollary 12). Thus a complete solution to Problem 1 is obtained.

Our second main result is that, under some cancellativity condition, an entropic algebra is a subreduct of a module over a commutative ring (Theorem 29). In fact the ring depends on cancellativity condition. If the condition is stronger,

then the ring has more invertible elements. For the standard cancellativity the ring is generated by invertible elements. Interestingly, Theorem 29 has a relatively easy proof if we additionally assume that considered algebra has an idempotent element (see the proof in case 2), while in the proof of general case we need to use Theorem 11.

The theory of entropic algebras is developed mostly for the groupoid case [12]. Modes were investigated in [24] and earlier in [22]. The most recent survey is [21] and the older ones are [20, 28].

Representations of entropic algebras have quite a long history. We finish this introduction with a sketch of it.

One of the first result about entropic algebras is the Bruck-Murdoch-Toyoda theorem [2, 17, 36] which says that for an entropic (in their terminology “abelian”) quasigroup (Q, \cdot) there is an abelian group $(Q, +, -, 0)$ with two commuting automorphisms f, g and an element $q \in Q$ such that $x \cdot y = f(x) + g(y) + q$. In particular, an entropic quasigroup is isotopic to an abelian group. T. Evans [1, 5] generalized the Bruck-Murdoch-Toyoda theorem and showed that for an entropic algebra (A, ω) with one n -ary basic operation ω and having certain regularity properties there is a commutative monoid $(A, +, 0)$ with pairwise commuting endomorphisms f_1, \dots, f_n and an element $a \in A$ such that $\omega(a_1, \dots, a_n) = f_1(a_1) + \dots + f_n(a_n) + a$ for $a_1, \dots, a_n \in A$. The conclusion seems to be considerably weaker than that of the Bruck-Murdoch theorem. However, if we assume that (A, ω) is an n -quasigroup, then the addition becomes a group operation and all homomorphisms f_1, \dots, f_n become automorphisms of $(A, +)$. Note that the above results for (n) -quasigroups have the following generalization [24, Corollary 6.2.6]: Each Mal’cev entropic algebra is polynomially equivalent to a module over a commutative ring. This is a consequence of the well known Smith-Gumm theorem [27, Theorem 418][8, Theorem 4.7]: A Mal’cev algebra is abelian (central in [27]) if and only if it is polynomially equivalent to a module.

A different kind of representation was given by J. Ježek and T. Kepka [11, 12]. They showed that for an entropic (this time authors use the name “medial”) groupoid (G, \cdot) satisfying $G \cdot G = G$ there is a commutative monoid $(M, +, 0)$ with two commuting automorphisms f, g such that $G \subseteq M$ and $x \cdot y = f(x) + g(y)$ for any $x, y \in G$. Therefore such groupoid is a subreduct of a semimodule over a commutative semiring with two invertible generators. This gives a positive solution to Problem 1 in the groupoid case. Note that Theorem 11 is a straightforward generalization of Ježek’s and Kepka’s result. It is worth explaining how the condition $G \cdot G = G$ appeared. J. Ježek and T. Kepka wanted to classify all simple entropic groupoids. The authors considered having a good representation theorem useful while working on this task. One may check that indeed all simple groupoids have surjective basic operation. However, according to our knowledge, the problem of classification of simple entropic groupoids is still open.

In [13] J. Ježek and T. Kepka generalized the above theorem and simplified the proof significantly. This generalization reaches far beyond our interest in this paper, but one can derive a corollary which is important for us. If an algebra with one surjective n -ary basic operation, where $n \geq 2$, is Szendrei and entropic, then it is a subreduct of a semimodule over a commutative semiring with n invertible generators. The technique used in the proof of our Theorem 11 is based on the

proof of this result presented in [13]. Some technical tricks are invented to deal with many basic operations, though.

Concerning Problem 1 for general modes, some work was done in [23, 24, 25, 26, 37, 38] by A. B. Romanowska, J. D. H. Smith and A. Zamojska-Dzienio. Their technique, though not universal, gave interesting results. For example, modes in the Mal'cev product of a quasivariety of cancellative Ω -modes and the variety of Ω -semilattices embed as subreducts into semimodules over certain commutative rings (Corollary 7.8.5 in [24]). More precisely, these semimodules are not semimodules in our sense because they do not satisfy the identity $0x = 0$ (see next Section). They are algebras in the regularization of the variety of modules over a certain commutative rings. A crucial tool used in the proof of this fact is the Romanowska-Smith theorem which says that each cancellative mode is a subreduct of an *affine space*, the full idempotent reduct of a module over a commutative ring (see [24, Section 7.7] and [23]). Our Theorem 29 is an extension of this result.

And lastly we would like to recall another interesting theorem due to J. Ježek and T. Kepka [12, Proposition 6.1.1]. A groupoid (G, \cdot) is a *division* groupoid if all its left and right multiplications are surjective, and (G, \cdot) is *regular* if it satisfies the quasi-identities $xu \approx xv \rightarrow yu \approx yv$ and $ux \approx vx \rightarrow uy \approx vy$. Note that abelian groupoids are regular. The theorem states that each entropic regular division groupoid is polynomially equivalent to a module over a commutative ring.

2. BASICS

We assume that the reader is familiar with basic algebraic concepts such as free algebras, satisfaction of identities, congruences, modules etc. [3, 15, 24]. By \mathbb{N} we denote the set of natural numbers. We fix a similarity type $\tau: \Omega \rightarrow \mathbb{N}$. When we consider algebras or terms not referring to their types we mean that their types coincide with τ . In particular, if we write “the variety of entropic algebras” we actually mean “the variety of entropic algebras of the type τ ”.

It will be convenient to fix an infinite countable set X of variables. We assume that X is disjoint from Ω . The set of terms with variables in X is denoted by $\text{Term}(X)$. The set of all variables occurring in a term t is denoted by $\text{arg}(t)$. A term t is *linear* if every variable in $\text{arg}(t)$ occurs in t exactly once. An identity $t \approx s$ is *linear* if both terms t, s are linear. An equational theory is *linear* if it has an equational basis consisting of linear identities.

A *valuation in algebra* (A, Ω) is a mapping $\mathbf{a}: X \rightarrow A; x \mapsto a_x$. By the universality property for the absolutely free algebra $(\text{Term}(X), \Omega)$, for a valuation \mathbf{a} in (A, Ω) , there is exactly one homomorphism $\tilde{\mathbf{a}}: (\text{Term}(X), \Omega) \rightarrow (A, \Omega)$ extending \mathbf{a} . For a term t we write $t(\mathbf{a})$ instead of $\tilde{\mathbf{a}}(t)$. A valuation in $(\text{Term}(X), \Omega)$ is called a *substitution*. If \mathbf{u} is a substitution, then $t(\mathbf{u})$ denotes the term obtained by substituting u_x for $x \in X$ in t .

We need basic information about semirings and semimodules [7, 9]. Semirings “are” rings without subtraction. Precisely, a *semiring* is an algebra $(S, +, 0, \cdot, 1)$ such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, the identities $0x = x0 = 0$ are true and multiplication distributes over addition. A semiring is *commutative* if its multiplication is commutative. The non-commutative semiring $(\mathbb{N}\langle V \rangle, +, 0, \cdot, 1)$ of polynomials with non-commuting indeterminants in V and natural coefficients is a free semiring over V . One may construct it as follows: first construct the free monoid $(V^*, \cdot, 1)$ over V , then construct the free commutative

monoid $(\mathbb{N}\langle V \rangle, +, 0)$ over V^* , and finally extend multiplication using $0x = x0 = 0$ and distributivity. Similarly, we may represent a free commutative semiring over V as the commutative semiring of polynomials with commuting indeterminates in V and natural coefficients. It is denoted by $(\mathbb{N}[V], +, 0, \cdot, 1)$. Let

$$\bar{} : (\mathbb{N}\langle V \rangle, +, 0, \cdot, 1) \rightarrow (\mathbb{N}[V], +, 0, \cdot, 1),$$

be the homomorphism extending the identity mapping on V . We will use it frequently, so it will be convenient to write $\bar{\alpha}$ instead of $\bar{}(\alpha)$ for $\alpha \in \mathbb{N}\langle V \rangle$.

By a *semimodule* over a semiring $(S, +, 0, \cdot, 1)$ we mean an algebra $(N, +, 0, S)$, where the unary operations determined by elements of S are endomorphisms of the commutative monoid $(N, +, 0)$ and moreover for $x \in N$, $s_1, s_2 \in S$

$$\begin{aligned} 1x &= x, \\ 0x &= 0, \\ (s_1 \cdot s_2)x &= s_1(s_2x), \\ (s_1 + s_2)x &= s_1x + s_2x. \end{aligned}$$

An element s of a commutative semiring $(S, +, 0, \cdot, 1)$ is *cancellable* if for all $r, r' \in S$ the equality $rs = r's$ implies $r = r'$, and is *invertible* if there is $r \in S$ such that $rs = 1$. In $(\mathbb{N}[V], +, 0, \cdot, 1)$ all nonzero elements are cancellable. We will use the following fact which may be proved as in the ring/module case.

Proposition 4. *Let $(S, +, 0, \cdot, 1)$ be a commutative semiring. Let U be a subset of S , closed under multiplication and consisting only of cancellable elements.*

- (1) *There exists a semiring of fractions $(U^{-1}S, +, 0, \cdot, 1)$ extending $(S, +, 0, \cdot, 1)$ in which every $u \in U$ is invertible.*
- (2) *If the basic operations of a semimodule $(K, +, 0, S)$ corresponding to the symbols from U are injective, then $(K, +, 0, S)$ is a subreduct of some semimodule $(N, +, 0, U^{-1}S)$.*

If in point (2) of Proposition 4 the operations corresponding to the symbols from U are bijective, then $(K, +, 0, S)$ is a reduct of the semimodule $(K, +, 0, U^{-1}S)$.

If $U = S - \{0\}$ then $(U^{-1}S, +, 0, \cdot, 1)$ is a semifield, that is a commutative semiring with all nonzero elements invertible.

3. STRONGLY ENTROPIC ALGEBRAS

Here strongly entropic algebras [32, 33] are recalled, and some of their basic properties are presented.

Let us distinguish some subsets of Ω . For a natural number i let

$$\begin{aligned} \Omega_i &= \{\omega \in \Omega \mid \tau(\omega) = i\}, \\ \Omega_{>i} &= \{\omega \in \Omega \mid \tau(\omega) > i\}. \end{aligned}$$

In particular, Ω_0 is the set of all nullary operation symbols and $\Omega_{>0}$ is the set of all non-nullary operation symbols. The *address* $a(t, y)$ of $y \in X \cup \Omega_0$ in a term t says how y is placed in t . The precise definition is as follows. Put

$$\Sigma = \{(\omega, i) \mid \omega \in \Omega_{>0} \text{ and } 1 \leq i \leq \tau(\omega)\}.$$

Let

$$a: \text{Term}(X) \times (X \cup \Omega_0) \rightarrow \mathbb{N}\langle \Sigma \rangle$$

be the function defined inductively by

$$a(x, y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

for $x, y \in X \cup \Omega_0$, and further by

$$a(\omega(t_1, \dots, t_{\tau(\omega)}), y) = \sum_{i=1}^{\tau(\omega)} (\omega, i) a(t_i, y)$$

for $\omega \in \Omega_{>0}$.

Put $\bar{a} = \bar{\circ} \circ a$. The set of identities $t \approx s$ satisfying the condition

$$(\forall x \in X) \bar{a}(t, x) = \bar{a}(s, x)$$

forms an equational theory. The variety corresponding to this theory is denoted by **SE** and its members are called *strongly entropic algebras*. Note that **SE** has a linear equational basis [33, Proposition 3.2.].

For a set Y we define $(N(Y), +, 0, \mathbb{N}[\Sigma])$ to be the semimodule over the semiring $(\mathbb{N}[\Sigma], +, 0, \cdot, 1)$ freely generated by the set Y . Elements of $N(Y)$ may be represented as sums $\sum_i \beta_i y_i$, where $\beta_i \in \Sigma^*$ and $y_i \in Y$. We equip the set $N(Y)$ with operations corresponding to symbols from Ω . Let

$$\omega(p_1, \dots, p_{\tau(\omega)}) = (\omega, 1)p_1 + \dots + (\omega, \tau(\omega))p_{\tau(\omega)}$$

for $\omega \in \Omega_{>0}$ and

$$o = 0$$

for $o \in \Omega_0$. By $(P(Y), \Omega)$ we denote the subalgebra of $(N(Y), \Omega)$ generated by Y .

Proposition 5 ([33, Proposition 3.1]). *The algebra $(N(Y), \Omega)$ belongs to **SE**. Moreover, $(P(Y), \Omega)$ is free in **SE** over Y .*

Let

$$b: \text{Term}(X) \times (X \cup \Omega_0) \rightarrow \mathbb{N}\langle \Omega_{>0} \rangle$$

and

$$\bar{b}: \text{Term}(X) \times (X \cup \Omega_0) \rightarrow \mathbb{N}[\Omega_{>0}]$$

be the mappings defined similarly to a and \bar{a} . Only replace each occurrence of (ω, i) by ω in the definitions. A term t is *isosceles* if there is a word $\gamma \in \Omega_{>0}^*$ such that for each $y \in X \cup \Omega_0$, $b(t, y) = k\gamma$ for some $k \in \mathbb{N}$. The word γ is called the *trace* of t . This condition says that the term t has a very regular form. All variables and all constants are on the same lowest level. On each level, except the lowest one, there is exactly one operation symbol from $\Omega_{>0}$.

Lemma 6. *Let s and t be linear isosceles terms. If an identity $s \approx t$ holds in **SE**, then it also holds in **SZE**.*

Proof. By entropicity we may assume that the traces of s and t coincide and that there are no symbols of nullary operations in t and s . Indeed, if there are such symbols we may assume that they are equal and substitute a new variable for them. Then

$$s(x_1, \dots, x_n) = t(x_{\pi(1)}, \dots, x_{\pi(n)}),$$

where π is a certain permutation of the set $\{1, \dots, n\}$ such that $\bar{a}(t, x_i) = \bar{a}(t, x_{\pi(i)})$. Each permutation is a composition of transpositions. Hence it is enough to consider the identities of the form

$$u(x_1, x_2, x_3, \dots, x_n) = u(x_2, x_1, x_3, \dots, x_n),$$

where u is a linear isosceles term with $a(u, x_1) = \alpha$, $a(u, x_2) = \beta$ and $\bar{\alpha} = \bar{\beta}$. Now we proceed by the induction on the depth of u . If the depth is two, then the assertion follows from Szendrei identities. If α and β have the same, say k -th letter, then by entropicity we may assume that they have the same first letter. In such case the assertion may be deduced by applying the inductive assumption to the appropriate subterm of u . We will show that the remaining cases may be reduced to this situation. So assume that the previous case does not hold and the depth of u is greater than two. Let $\omega_1 \dots \omega_m$, $m > 2$, be the trace of u .

Case 1: $\omega_k \neq \omega_1$ for some k . Then α must be of the form $(\omega_1, i)\alpha'(\omega_k, l)\alpha''$ and β must be of the form $(\omega_1, j)\beta'(\omega_k, l)\beta''$, where $i \neq j$. The reduction may be done by applying entropicity to the subterm of u under the address (ω_1, i) (or (ω_1, j)).

Case 2: $\omega_1 = \dots = \omega_m = \omega$. Assume that $\alpha = (\omega, i)(\omega, j)\alpha'$. If $\beta = (\omega, k)\beta'$, where $k \neq j$, then we may apply entropicity to the subterm of u under the address (ω, k) . So assume that $\beta = (\omega, j)(\omega, k)\beta'$. If $k \neq i$, then we may proceed as previously. Finally, in the remaining case, $\alpha = (\omega, i)(\omega, j)(\omega, k)\alpha'$ and $\beta = (\omega, j)(\omega, i)\beta'(\omega, k)\beta''$. Here the assumption that the depth of u is greater than two comes out. Then we may do the reduction by applying entropicity to the subterm of u under the address $(\omega, j)(\omega, i)$. \square

Lemma 7 ([33, Corollary 4.3]). *Let t be a linear term such that the equality $\bar{b}(t, x) = \bar{b}(t, y)$ is valid for all $x, y \in \arg(t)$. Then there is an isosceles linear term t' such that the identity $t \approx t'$ holds in \mathbf{E} .*

Lemma 8. *For each linear term t there are an isosceles linear term t' and a substitution \mathbf{u} of $\Omega_{>0}$ -terms such that the identity $t(\mathbf{u}) \approx t'$ holds in \mathbf{E} .*

Proof. Let $\mathbf{u}: x \mapsto u_x$ be a substitution such that u_x is an isosceles linear $\Omega_{>0}$ -term with the trace

$$\prod_{\substack{y \in \arg(t) \\ y \neq x}} b(t, y).$$

Moreover, we assume that $\arg(u_x) \cap \arg(u_y) = \emptyset$ for distinct $x, y \in X$. Then the term $t(\mathbf{u})$ is linear. For all $x, y \in \arg(t)$ we have

$$\bar{b}(t(\mathbf{u}), x) = \bar{b}(t(\mathbf{u}), y).$$

Hence, by Lemma 7, there is a term t' with desired properties. \square

The following notation will be very useful in the next section, but we also use it in the proof of the next proposition. For a term t , an algebra (A, Ω) and its element a , let $\mathbf{a}^t: X \rightarrow A$ be a valuation such that $t(\mathbf{a}^t) = a$. Obviously such a valuation neither has to exist nor has to be unique.

Proposition 9. *Let (A, Ω) be a Szendrei entropic algebra with surjective basic non-nullary operations. Then (A, Ω) is strongly entropic.*

Proof. Let $s \approx t$ be a linear identity such that $\bar{a}(s, x) = \bar{a}(t, x)$ for all $x \in X$. We will show that (A, Ω) satisfies $s \approx t$. Applying Lemma 8 to the term s we obtain a linear isosceles term s' and a substitution \mathbf{u} such that the identity $s(\mathbf{u}) \approx s'$ holds in \mathbf{E} . Note that the term $t(\mathbf{u})$ satisfies the condition of Lemma 7. Hence there is a linear isosceles term t' such that the identity $t(\mathbf{u}) \approx t'$ holds in \mathbf{E} . The identity $s' \approx t'$ holds in \mathbf{SE} . Thus, by Lemma 6, it holds in \mathbf{SZE} .

Let $\mathbf{a}: X \rightarrow A$; $x \mapsto a_x$ be a valuation and put $b_y = (a_x)_{y^x}$ for $y \in \arg(u_x)$. In particular, this means that $u_x(\mathbf{b}) = u_x(\mathbf{a}^{u_x}) = a_x$. Note that the existence of the valuations \mathbf{a}^{u_x} is ensured by the assumption from the proposition and the fact that u_x are $\Omega_{>0}$ -terms. Then

$$s(\mathbf{a}) = s(\mathbf{u})(\mathbf{b}) = s'(\mathbf{b}) = t'(\mathbf{b}) = t(\mathbf{u})(\mathbf{b}) = t(\mathbf{a}).$$

So the identity $s \approx t$ holds in (A, Ω) . \square

The reader may check that Proposition 9 may be slightly strengthened by assuming only the surjectivity of the basic operations of arity at least 2. However, we will not make a use of this fact.

Corollary 10. *Szendrei modes are strongly entropic.*

4. SUBREDUCTS OF SEMIMODULES

Consider the semiring of fractions

$$(T, +, 0, \cdot, 1) = ((\Sigma_{>1})^*)^{-1}\mathbb{N}[\Sigma], +, 0, \cdot, 1,$$

where

$$\Sigma_{>1} = \{(\omega, i) \mid \omega \in \Omega_{>1} \text{ and } 1 \leq i \leq \tau(\omega)\} = \Sigma - \Omega_1 \times \{1\}.$$

This section is devoted to the proof of the following theorem.

Theorem 11. *Let (A, Ω) be a Szendrei entropic algebra without constant basic operations such that $\omega(A, \dots, A) = A$ for every $\omega \in \Omega_{>1}$. There exists a semimodule $(N, +, 0, T)$ over the commutative semiring $(T, +, 0, \cdot, 1)$ such that for $\omega \in \Omega$ and $a_1, \dots, a_{\tau(\omega)} \in A$ we have*

$$\omega(a_1, \dots, a_{\tau(\omega)}) = (\omega, 1)a_1 + \dots + (\omega, \tau(\omega))a_{\tau(\omega)}.$$

We readily obtain the following consequence of Theorem 11.

Corollary 12. *Every Szendrei mode is a subreduct of a semimodule over a commutative semiring.*

We start with considering a fixed Szendrei entropic algebra (A, Ω) . We assume that $\Omega_0 = \emptyset$ and $\omega(A, \dots, A) = A$ for all $\omega \in \Omega$.

Let $(N(A), +, 0, \mathbb{N}[\Sigma])$, $(N(A), \Omega)$ and $(P(A), \Omega)$ be algebras defined as in previous section, where $Y = A$.

We introduce one more notation. It simplifies coding the proof and, we hope, also reading it. Because $A \subseteq N(A)$, the symbol $t(\mathbf{b})$, where t is a term and \mathbf{b} is a valuation in (A, Ω) , may be interpreted in two different ways: as the result of the term operation in (A, Ω) , or as the result of the term operation in $(N(A), \Omega)$, that is $\sum_{x \in \arg(t)} \bar{a}(t, x)b_x$. In order to distinguish these two cases we will use symbol $t_N(\mathbf{b})$ for the latter one. Note that, because $(P(A), \Omega)$ is a subalgebra of $(N(A), \Omega)$, we do not need to introduce a special notation for term operations in $(P(A), \Omega)$.

On the set $N(A)$ we define a binary relations \searrow as follows. For $p, q \in N(A)$ let $p \searrow q$ if there are $w \in N(A)$, $\gamma \in \Sigma^*$, an element $c \in A$, a term t , and a valuation \mathbf{c}^t such that

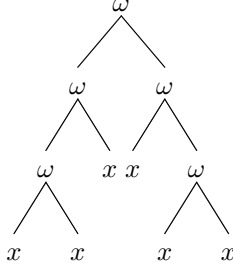
$$p = w + \bar{\gamma}c \quad \text{and} \quad q = w + \bar{\gamma}t_N(\mathbf{c}^t).$$

Recall that $\mathbf{c}^t: x \mapsto c_x^t$ is a valuation satisfying $t(\mathbf{c}^t) = c$. Let \nearrow be the inverse of \searrow . Note that \nearrow and \searrow are reflexive and closed under unary polynomials of $(N(A), +, \mathbb{N}[\Sigma])$. Thus

$$\Theta = (\nearrow \cup \searrow)^\infty,$$

where R^∞ denotes the transitive closure of R , is a congruence of $(N(A), +, 0, \mathbb{N}[\Sigma])$ and of $(N(A), \Omega)$. Our aim is to show that $(A, \Omega) \rightarrow (N(A), \Omega)/\Theta$; $a \mapsto a/\Theta$ is an injective homomorphism. It is the core of the proof of Theorem 11. It will be done in a series of lemmas.

Observe that $p \in P(A)$ if and only if there are a (linear) term t and a valuation \mathbf{b} such that $p = t_N(\mathbf{b})$. Note that if $p \in P(A)$ and $p \searrow q$, then $q \in P(A)$. The converse does not have to hold. Indeed, consider a binary term $t(x) = \omega(\omega(\omega(x, x), x), \omega(x, \omega(x, x)))$. Its parsing tree is



and for $a \in A$ we have

$$\begin{aligned} P(A) \ni t_N(a) &= [(\omega, 1)(\omega, 1)(\omega, 1) + (\omega, 1)(\omega, 2) + (\omega, 2)(\omega, 1) + (\omega, 2)(\omega, 2)(\omega, 2)]a \\ &\quad + (\omega, 1)(\omega, 2)[(\omega, 1) + (\omega, 2)]a \\ &\nearrow [(\omega, 1)(\omega, 1)(\omega, 1) + (\omega, 1)(\omega, 2) + (\omega, 2)(\omega, 1) + (\omega, 2)(\omega, 2)(\omega, 2)]a \\ &\quad + (\omega, 1)(\omega, 2)b \notin P(A), \end{aligned}$$

where $\omega(a, a) = b$.

Lemma 13. *Let t be a linear isosceles term, s be a term and $\gamma \in \Sigma^*$ be a word such that $\bar{a}(t, x) = \bar{\gamma}\bar{a}(s, x)$ for all $x \in \arg(s)$. Then there exists a term t' such that $t \approx t'$ holds in **SE** (also in **SZE**) and s is a subterm of t' under the address γ .*

Proof. By Lemma 7 we may assume that s is an isosceles term with the trace δ . We may take t' as an isosceles term with the trace $\gamma\delta$, where $a(t', x) = \gamma a(s, x)$ for all $x \in \arg(s)$ and $\bar{a}(t', y) = \bar{a}(t, y)$ for all $y \in \arg(t)$. Obviously, $t \approx t'$ holds in **SE** and, by Lemma 6, it also holds in **SZE**. \square

The importance of isosceles terms follows from the next lemma.

Lemma 14. *Let $p = t_N(\mathbf{b}) \in P(A)$ for some linear isosceles term t and some valuation \mathbf{b} . If $p \nearrow q$, then $q \in P(A)$.*

Proof. Let

$$p = w + \bar{\gamma}s_N(\mathbf{c}^s) \quad \text{and} \quad q = w + \bar{\gamma}c,$$

where $w \in N(A)$, $\gamma \in \Sigma^*$, s is a linear term and $c \in A$. For $x \in \arg(s)$ let $x' \in \arg(t)$ be a variable such that

$$\bar{a}(t, x')b_{x'} = \bar{\gamma}\bar{a}(s, x)c_x^s.$$

Let s' be a term obtained from s by replacing each variable $x \in \arg(s)$ by x' . Then $s'(\mathbf{b}) = c$ and $q = w + \bar{\gamma}s'_N(\mathbf{b})$. Terms t and s' satisfy the condition from Lemma 13. Thus there exists a linear isosceles term t' such that s' is its subterm under the address γ and the identity $t \approx t'$ holds in \mathbf{SE} . Thus, by Proposition 5, $p = t'_N(\mathbf{b})$. Let t'' be a term obtained from t' by putting a new variable y , $y \notin \arg(t')$, under the address γ . Let $b'_x = b_x$ for $x \neq y$ and $b'_y = s'(\mathbf{b}) = c$. Then $q = t''_N(\mathbf{b}') \in P(A)$. \square

Lemma 15. *Let $p \nearrow q \searrow r$. If $p \in P(A)$, then there are $p', q', r' \in P(A)$ such that*

$$p \searrow^\infty p' \nearrow q' \searrow r' \nearrow^\infty r.$$

Proof. Let $p = t_N(\mathbf{b})$, where t is a linear term. There are two cases.

Case 1:

$$\begin{aligned} p &= w + \bar{\gamma}s_N(\mathbf{c}^s), \\ q &= w + \bar{\gamma}c, \\ r &= w + \bar{\gamma}v_N(\mathbf{c}^v). \end{aligned}$$

By the reasoning from the beginning of the proof of Lemma 14, we may assume that $\mathbf{c}^s = \mathbf{b}$. By Lemma 8, there are an isosceles linear term t' and a substitution \mathbf{u} such that the identity $t' \approx t(\mathbf{u})$ holds in \mathbf{E} . For each $x \in X$ choose $\mathbf{b}_x^{u_x}$. The existence of these valuations is guaranteed by the assumption that all basic operations of (A, Ω) are surjective. Let \mathbf{b}' be a valuation such that $b'_y = (b_x^{u_x})_y$ for $y \in \arg(u_x)$ (note that if $x_1 \neq x_2$, then $\arg(u_{x_1}) \cap \arg(u_{x_2}) = \emptyset$). Define

$$\begin{aligned} p' &= t(\mathbf{u})_N(\mathbf{b}') = t'_N(\mathbf{b}') = w' + \bar{\gamma}s(\mathbf{u})_N(\mathbf{b}'), \\ q' &= w' + \bar{\gamma}c, \\ r' &= w' + \bar{\gamma}v_N(\mathbf{c}^v). \end{aligned}$$

Here w' is the appropriate element of $N(A)$.

We have $p \searrow^\infty p' \nearrow q' \searrow r'$ and $p' \in P(A)$. By Lemma 14, $q' \in P(A)$, and thus $r' \in P(A)$. It remains to show that $r' \nearrow^\infty r$. It follows from the following computation

$$\begin{aligned} w' &= \sum_{\substack{y \in \arg(t(\mathbf{u})) \\ y \notin \arg(s(\mathbf{u}))}} \bar{a}(t(\mathbf{u}), y)b'_y = \sum_{\substack{x \in \arg(t) \\ x \notin \arg(s)}} \bar{a}(t, x)(u_x)_N(\mathbf{b}') \\ &= \sum_{\substack{x \in \arg(t) \\ x \notin \arg(s)}} \bar{a}(t, x)(u_x)_N(\mathbf{b}_x^{u_x}) \nearrow^\infty \sum_{\substack{x \in \arg(t) \\ x \notin \arg(s)}} \bar{a}(t, x)b_x = w. \end{aligned}$$

Case 2:

$$\begin{aligned} p &= w + \bar{\gamma}s_N(\mathbf{c}^s) + \bar{\delta}d, \\ q &= w + \bar{\gamma}c + \bar{\delta}d, \\ r &= w + \bar{\gamma}c + \bar{\delta}v_N(\mathbf{d}^v). \end{aligned}$$

We may take

$$p' = q' = r' = w + \bar{\gamma}s_N(\mathbf{c}^s) + \bar{\delta}v_N(\mathbf{d}^v).$$

\square

Lemma 16. $\nearrow \circ \nearrow \circ \searrow \subseteq \nearrow \circ \searrow \circ \nearrow$ and $\nearrow \circ \searrow \circ \searrow \subseteq \searrow \circ \nearrow \circ \searrow$.

Proof. It is enough to prove the first inclusion. Let

$$p \nearrow \circ \nearrow q \searrow r.$$

We may assume that $p \nearrow q$ does not hold. Then

$$\begin{aligned} p &= w + \bar{\gamma}s_N(\mathbf{c}^s) + \bar{\delta}t_N(\mathbf{d}^t) \quad \text{and} \\ q &= w + \bar{\gamma}c + \bar{\delta}d. \end{aligned}$$

Now either $r = w' + \bar{\gamma}c + \bar{\delta}d$, where $w \searrow w'$, and then

$$p \nearrow w + \bar{\gamma}s_N(\mathbf{c}^s) + \bar{\delta}d \searrow w' + \bar{\gamma}s_N(\mathbf{c}^s) + \bar{\delta}d \nearrow r,$$

or $r = w + \bar{\gamma}c + \bar{\delta}v_N(\mathbf{d}^v)$ and then

$$p \nearrow w + \bar{\gamma}s_N(\mathbf{c}^s) + \bar{\delta}d \searrow w + \bar{\gamma}s_N(\mathbf{c}^s) + \bar{\delta}v_N(\mathbf{d}^v) \nearrow r.$$

□

Put

$$\Psi = ((\nearrow \cup \searrow) \cap P(A)^2)^\infty.$$

Lemma 17. If $p \in P(A)$, $q \in N(A)$ and $p \Theta q$, then there is $q' \in P(A)$ such that

$$p \Psi q' \nearrow^\infty q.$$

Proof. By Lemma 16, there exists a natural number k such that

$$p \searrow^\infty \circ (\nearrow \circ \searrow)^k \circ \nearrow^\infty q.$$

We proceed by induction on k . If $k = 0$ the assertion is obvious. Assume that $k > 0$ and that the assertion is true for $k - 1$. Let p_1, p_2, p_3 be such that

$$p \searrow^\infty p_1 \nearrow p_2 \searrow p_3 (\nearrow \circ \searrow)^{k-1} \circ \nearrow^\infty q.$$

By Lemma 15, there are $p'_1, p'_2, p'_3 \in P(A)$ such that

$$p \searrow^\infty p'_1 \nearrow p'_2 \searrow p'_3 \nearrow^\infty p_3 (\nearrow \circ \searrow)^{k-1} \circ \nearrow^\infty q.$$

By Lemma 16

$$p'_3 (\nearrow \circ \searrow)^{k-1} \circ \nearrow^\infty q.$$

By the inductive assumption there exists $q' \in P(A)$ such that

$$p'_3 \Psi q' \nearrow^\infty q.$$

Finally $p \Psi p'_3$ and the transitivity of Ψ yields $p \Psi q'$. □

Lemma 18. $\Psi = \Theta \cap P(A)^2$.

Proof. Let $p, q \in P(A)$ and $p \Theta q$. By Lemma 17, there exists q' such that $p \Psi q' \nearrow^\infty q$. But $q, q' \in P(A)$ yields $(q, q') \in \Psi$. Hence $p \Psi q$. The converse inclusion is evident. □

Let id_A denotes the equality relation on the set A .

Lemma 19. $\Theta \cap A^2 = \text{id}_A$.

Proof. By Proposition 5, the algebra $(P(A), \Omega)$ is free in \mathbf{SE} over the set A . Moreover, by Proposition 9, the algebra (A, Ω) belongs to \mathbf{SE} . Hence there exists the unique homomorphism $\pi: (P(A), \Omega) \rightarrow (A, \Omega)$ such that $\pi(a) = a$ for $a \in A$. More precisely π is given by the assignment

$$\pi: t_N(\mathbf{a}) \mapsto t(\mathbf{a}).$$

Note that

$$\setminus \cap P(A)^2 \subseteq \ker(\pi).$$

Thus

$$\Psi \subseteq \ker(\pi).$$

Let $a, b \in A$. By Lemma 18, if $a \Theta b$, then $a \Psi b$. Thus $a = \pi(a) = \pi(b) = b$. \square

Lemma 20. *If $(\omega, k)p \Theta (\omega, k)q$, where $\omega \in \Omega_{>1}$, then $p \Theta q$.*

Proof. First we prove that if $(\omega, k)p \Theta r$, then there exists r' such that $r = (\omega, k)r'$ and $p \Theta r'$. If $(\omega, k)p \setminus r$ then the statement is evident. Assume now that

$$(\omega, k)p = s + \bar{\beta} \sum_{i=1}^{\tau(\mu)} (\mu, i)a_i \quad \text{and} \quad r = s + \bar{\beta}a,$$

where $\mu(a_1, \dots, a_{\tau(\mu)}) = a$ in (A, Ω) . If $\mu \neq \omega$ then $\bar{\beta} = (\omega, k)\bar{\beta}'$, $s = (\omega, k)s'$ and $p = s' + \bar{\beta}' \sum (\mu, i)a_i$. Thus we may put $r' = s' + \bar{\beta}'a$. Now let $\mu = \omega$. We assumed that the arity of ω must be greater than one. Therefore there exists a natural number $l \neq k$ such that

$$(\omega, k)p = s + \bar{\beta} \sum_{\substack{i=1 \\ i \neq l}}^{\tau(\omega)} (\omega, i)a_i + \bar{\beta}(\omega, l)a_l.$$

Again $\bar{\beta} = (\omega, k)\bar{\beta}'$, $s = (\omega, k)s'$ and $r' = s' + \bar{\beta}'a$. Now the assertion follows by induction.

Now if $(\omega, k)p \Theta (\omega, k)q$, then $(p, q') \in \Theta$ and $(\omega, k)q = (\omega, k)q'$ for some q' . But then necessarily $q = q'$. \square

The results obtained till now in this section easily give the conclusion of Theorem 11 for algebra (A, Ω) . Now we show that the assumption that operations corresponding to symbols from Ω_1 are onto is irrelevant. So let (A, Ω) be an algebra satisfying the conditions from Theorem 11. Obviously all obtained results may be applied to the reduct $(A, \Omega_{>1})$. Let $(N_{>1}(A), +, 0, \mathbb{N}[\Sigma_{>1}])$ and $\Theta_{>1}$ be analogs of $(N(A), +, 0, \mathbb{N}[\Sigma_{>1}])$ and Θ respectively defined for $(A, \Omega_{>1})$. For every $\mu \in \Omega_1$ define the corresponding operation on $N_{>1}(A)$ by

$$\mu \left(\sum \tilde{\gamma}_i a_i \right) = \sum \tilde{\gamma}_i \mu(a_i),$$

where $\beta_i \in \Sigma_{>1}^*$, $a_i \in A$.

Lemma 21. *The relation $\Theta_{>1}$ is a congruence of $(N_{>1}(A), +, 0, \mathbb{N}[\Sigma_{>1}], \Omega_1)$. Moreover, the algebra $(N_{>1}(A), +, 0, \mathbb{N}[\Sigma_{>1}], \Omega_1) / \Theta_{>1}$ is term equivalent to a semimodule $(N_{>1}(A), +, 0, \mathbb{N}[\Sigma])$.*

Proof. For the first statement it is enough to show that if $p, q \in N_{>1}(A)$ and $p \searrow q$, then $\mu(p) \searrow \mu(q)$. Consider a valuation \mathbf{b}^t in $(A, \Omega_{>1})$ and let $\mu(\mathbf{b}^t)$ be a valuation given by $x \mapsto \mu(b_x^t)$. By entropy $t(\mu(\mathbf{b}^t)) = \mu(b)$, and thus

$$\mu(b) \searrow t_N(\mu(\mathbf{b}^t)) = \mu(t_N(\mathbf{b}^t)).$$

The general case follows from the fact that μ is an endomorphism of the semimodule $(N_{>1}(A), +, 0, \mathbb{N}[\Sigma_{>1}])$. The second statement is evident. \square

Proof of Theorem 11. By Lemmas 19 and 21, algebra (A, Ω) is a subreduct of the semimodule $(N_{>1}(A), +, 0, \mathbb{N}[\Sigma])/\Theta$. Moreover, this semimodule, by Proposition 4 and by Lemma 20, is a subreduct of a semimodule $(N, +, 0, T)$. \square

We end this section with a couple of comments.

Remark 22. Let (A, Ω) be any algebra without constants. Then (A, Ω) is a subreduct of a semimodule over a commutative semiring if and only if $\Theta \cap A^2 = \text{id}_A$. In Lemma 20 we did not use the assumption that basic operations are onto. Thus (A, Ω) is a subreduct of a semimodule over a commutative semiring if and only if it is a subreduct of a semimodule over $(T, +, 0, \cdot, 1)$.

Remark 23. Note that $(N, +, 0, T)$ trivially satisfies the condition $N + N = N$. Thus for $p \in N$ and $\omega \in \Omega_{>1}$ we have

$$\begin{aligned} p &= p_1 + \cdots + p_{\tau(\omega)} \\ &= (\omega, 1)(\omega, 1)^{-1}p_1 + \cdots + (\omega, \tau(\omega))(\omega, \tau(\omega))^{-1}p_{\tau(\omega)} \\ &= (\omega, 1)q_1 + \cdots + (\omega, \tau(\omega))q_{\tau(\omega)} \\ &= \omega(q_1, \dots, q_{\tau(\omega)}) \end{aligned}$$

for some $p_i \in N$ and $q_i = (\omega, i)^{-1}p_i$. Thus, a Szendrei entropic algebra is a subreduct of a semimodule over a commutative semiring if and only if it is a subalgebra of a Szendrei entropic algebra (B, Ω) satisfying $\omega(B, \dots, B) = B$ for $\omega \in \Omega_{>1}$.

Remark 24. Theorem 11 is no longer true if we allow constants to appear in the structure of a considered algebra. Let $(\{a, b\}, \vee, b)$ be the algebra, where $(\{a, b\}, \vee)$ is a semilattice in which $a < b$. Assume that it is a subreduct of a semimodule $(M, +, 0, S)$. Then there are $s, r \in S$ such that $x \vee y = rx + sy$ for $x, y \in \{a, b\}$, and $b = 0$. So

$$a = a + b = (ra + sa) + (rb + sb) = (ra + sb) + (rb + sa) = b + b = b,$$

a contradiction.

Remark 25. It is not true that each strongly entropic algebra without constants and with at least one non-unary surjective basic operation is a subreduct of a semimodule over a commutative semiring. Let (A, \diamond, \circ) be an algebra, where (A, \diamond) is a strongly entropic groupoid which is not a subreduct of a semimodule over a commutative semiring [12, Example 3.2.1.] and $x \circ y = x$. Obviously (A, \diamond, \circ) is not a subreduct of a semimodule over a commutative semiring. With a $\{\diamond, \circ\}$ -term t we associate a $\{\diamond\}$ -term t^* given inductively by the rule: $x^* = x$ for $x \in X$, $t^* = t_1^*$ if $t = t_1 \circ t_2$, and $t^* = t_1^* \diamond t_2^*$ if $t = t_1 \diamond t_2$. Consider an identity $s \approx t$, where $\bar{a}(s, x) = \bar{a}(t, x)$ for $x \in X$. Then $\bar{a}(s^*, x) = \bar{a}(t^*, x)$ for $x \in X$. Thus, for a valuation $\mathbf{a}: X \rightarrow A$, we obtain

$$s(\mathbf{a}) = s^*(\mathbf{a}) = t^*(\mathbf{a}) = t(\mathbf{a}).$$

Hence (A, \diamond, \circ) is strongly entropic.

Remark 26. Assume that an algebra (A, Ω) satisfies the conditions from Theorem 11. With a set $\{u_i(x) \approx v_i(x) \mid i \in I\}$ of identities with one variable we associate the congruence α of the semiring $(T, +, 0, \cdot, 1)$ generated by the set $\{\bar{a}(u_i, x), \bar{a}(v_i, x) \mid i \in I\}$. If (A, Ω) satisfies $u_i \approx v_i$ for all $i \in I$, then it is a subreduct of a semimodule over $(T, +, 0, \cdot, 1)/\alpha$. Indeed, if $p = \sum \lambda_k a_k \in N$, where $(N, +, 0, T)$ is the semimodule from Theorem 11, then

$$\bar{a}(u_i, x)p = \sum_k \lambda_k \bar{a}(u_i, x)a_k = \sum_k \lambda_k b_k^i = \sum_k \lambda_k \bar{a}(v_i, x)a_k = \bar{a}(v_i, x)p.$$

Here $b_k^i = u_i(a_k) = v_i(a_k)$. This fact may be in particular applied to Szendrei modes for idempotent identities.

Remark 27. After presenting the proof of Theorem 11 in [32], D. Stanovský simplified the proof of Corollary 12 [30]. This simplification is based on a more essential use of idempotency.

5. CANCELLATIVITY

We need to recall the notions of \mathcal{M} -cancellativity and \mathcal{M} -polyquasigroup [33]. We will use them in the next section.

We fix one variable from the set X and denote it by v . The set $\text{Term}(X)$ has a monoid structure, where multiplication is given by

$$t(v, x_1, \dots, x_m) \cdot s(v, y_1, \dots, y_n) = t(s(v, y_1, \dots, y_n), x_1, \dots, x_m)$$

and v is its neutral element.

Let \mathcal{S} be the set of terms in $\text{Term}(X)$ such that v occurs in them, that is

$$\mathcal{S} = \{t \in \text{Term}(X) \mid \mathfrak{a}(t, v) \neq 0\}.$$

Note that $(\mathcal{S}, \cdot, v) \leq (\text{Term}(X), \cdot, v)$.

By a *monoid of terms* we mean a subset \mathcal{M} of $\text{Term}(X)$ satisfying

- (M1) (\mathcal{M}, \cdot, v) is a submonoid of (\mathcal{S}, \cdot, v) ;
- (M2) \mathcal{M} is closed under substitution: if $t(v, x_1, \dots, x_n) \in \mathcal{M}$ and $t_1, \dots, t_n \in \text{Term}(X) - \mathcal{S}$, then $t(v, t_1, \dots, t_n) \in \mathcal{M}$.

Let \mathcal{L} be the set of all terms in $\text{Term}(X)$ such that v occurs in them exactly once and \mathcal{P} be the set of all terms in which v occurs exactly once but always on the rightmost place. Sets \mathcal{S} , \mathcal{L} and \mathcal{P} are examples of monoids of terms.

We need to consider two more conditions.

- (P) if $\Omega_{>1} \neq \emptyset$ then there exist a term $\eta(x_1, x_2, x_3)$ and distinct variables $y, z \neq v$ such that $\eta(v, y, z), \eta(y, v, z) \in \mathcal{M}$.
- (Ax) For each $\omega \in \Omega_{>1}$ there are an integer $1 \leq i \leq \tau(\omega)$ and a variable $z \in X$ such that

$$\omega(z, \dots, z, v, z, \dots, z) \in \mathcal{M},$$

where v occurs in the i -th slot;

Note that x_1, x_2 have to occur in η while x_3 does not. A monoid \mathcal{M} of terms is *proper* provided it satisfies condition (P). Monoids of terms \mathcal{S} and \mathcal{L} are proper, and \mathcal{P} is proper only if $\Omega_{>1} = \emptyset$. All $\mathcal{S}, \mathcal{L}, \mathcal{P}$ satisfy (Ax).

A *translation* of an algebra (A, Ω) is a mapping

$$s(-, a_1, \dots, a_m): A \rightarrow A; x \mapsto s(x, a_1, \dots, a_m),$$

where $s(v, x_1, \dots, x_m) \in \text{Term}(X)$ and $a_1, \dots, a_m \in A$. A mapping $f: A \rightarrow A$ is an \mathcal{M} -translation if there are a term $s(v, x_1, \dots, x_m) \in \mathcal{M}$ and elements $a_1, \dots, a_m \in A$ such that $f = s(-, a_1, \dots, a_m)$.

An algebra is \mathcal{M} -cancellative if all its \mathcal{M} -translations are injective. For instance an algebra is \mathcal{L} -cancellative if and only if it is cancellative. An algebra is called an \mathcal{M} -polyquasigroup if all its \mathcal{M} -translations are bijective.

The following theorem summarizes important for us facts obtained in [33, Theorem 7.1 and Theorem 9.3].

Theorem 28. *Let \mathcal{M} be a monoid of terms. Then*

- (1) *every \mathcal{M} -cancellative strongly entropic algebra embeds into a strongly entropic \mathcal{M} -polyquasigroup;*
- (2) *if \mathcal{M} is proper then each entropic \mathcal{M} -cancellative algebra is strongly entropic and hence satisfies Szendrei identities.*

Note that Proposition 4 may be deduced from Theorem 28 point (1).

6. SUBREDUCTS OF MODULES

In this section we will show that for a proper monoid \mathcal{M} of terms satisfying (Ax) \mathcal{M} -cancellative entropic algebras are subreducts of modules over commutative rings.

Firstly let us specify what kind of rings we are considering. For a monoid \mathcal{M} of terms let

$$\mathcal{M}_v = \bar{a}(\mathcal{M}, v).$$

The set \mathcal{M}_v may be considered as the subset of the commutative ring of polynomials $(\mathbb{Z}[\Sigma], +, 0, \cdot, 1)$ with indeterminants in Σ and integer coefficients. Note that $(\mathbb{Z}[\Sigma], +, 0, \cdot, 1)$ is an integral domain and \mathcal{M}_v is closed under multiplication. Define

$$(R^{\mathcal{M}}, +, -, 0, \cdot, 1) = (\mathcal{M}_v^{-1}\mathbb{Z}[\Sigma], +, -, 0, \cdot, 1)$$

the ring of fractions of $(\mathbb{Z}[\Sigma], +, -, 0, \cdot, 1)$ with respect to the set \mathcal{M}_v .

Theorem 29. *Let \mathcal{M} be a proper monoid of terms satisfying (Ax). If (A, Ω) is an \mathcal{M} -cancellative entropic algebra, then there exists a module $(M, +, -, 0, R^{\mathcal{M}})$ such that for all $\omega \in \Omega$ and $a_1, \dots, a_{\tau(\omega)} \in A$ we have*

$$\omega(a_1, \dots, a_{\tau(\omega)}) = (\omega, 1)a_1 + \dots + (\omega, \tau(\omega))a_{\tau(\omega)}.$$

Proof. The monoid \mathcal{M} of terms is proper. Hence, by Theorem 28, the algebra (A, Ω) embeds into an \mathcal{M} -polyquasigroup. Thus in what follows we may assume that A is already an \mathcal{M} -polyquasigroup. The proof splits into three cases:

Case 1: $\Omega_{>1} = \emptyset$

Let $(M, +, -, 0)$ be the free abelian group over the set $A - \Omega_0$. Here we consider Ω_0 as a subset of A . Equivalently, $(M, +, -, 0, \mathbb{Z})$ is the free module over the ring of integers with basis $A - \Omega_0$. We equip the set M with an Ω -structure by defining

$$\omega\left(\sum \xi_j a_j\right) = \sum \xi_j \omega(a_j) \quad \text{and} \quad o = 0,$$

where $\xi_j \in \mathbb{Z}$, $a_j \in A$ and $\omega \in \Omega_1$, $o \in \Omega_0$. Note that the set M has a natural structure of a module over the ring $(\mathbb{Z}[\Sigma], +, -, 0, \cdot, 1)$, where the operation determined by a scalar $(\omega, 1)$ coincides with ω . In order to finish the proof in this case it is enough to realize that for $t(v) \in \mathcal{M}$ the operation $\bar{a}(t, v)$ is bijective on M . It follows from the fact that (A, Ω) is an \mathcal{M} -polyquasigroup, and thus the term operation given by t is bijective on A , and hence on M .

Case 2: $\Omega_{>1} \neq \emptyset$ and $\Omega_0 \neq \emptyset$

The proof goes in a standard fashion (see [5] or [24, Section 7.7]). As \mathcal{M} is proper, there exist a term $\eta(x_1, x_2, x_3)$ and distinct variables $y, z \neq v$ such that $\eta(v, y, z), \eta(y, v, z) \in \mathcal{M}$. Because (A, Ω) is an entropic \mathcal{M} -polyquasigroup we have two isomorphisms

$$\begin{aligned} f: (A, \Omega) &\rightarrow (A, \Omega); a \mapsto \eta(a, o, o), \\ g: (A, \Omega) &\rightarrow (A, \Omega); a \mapsto \eta(o, a, o), \end{aligned}$$

where o is a constant operation. Put

$$b + c = t(f^{-1}(b), g^{-1}(c)).$$

That provides a structure of an abelian group $(A, +, -, 0)$ with o as a neutral element. Indeed, we have $(a+b)+(c+d) = (a+c)+(b+d)$ and $a+o = o+a = a$ for all $a, b, c, d \in A$. It readily yields the associativity and commutativity of $+$. Moreover, because $\omega(-, -, o)$ is a quasigroup operation, $+$ is an abelian group operation. To get the structure of a module over the ring $(\mathbb{Z}[\Sigma], +, -, 0, \cdot, 1)$ first define

$$(\mu, k)a = \mu(0, \dots, 0, a, 0, \dots, 0),$$

where a is in the k -th slot, and next extend this definition in a natural way to obtain operations corresponding to symbols from $\mathbb{Z}[\Sigma]$. Algebras $(A, \Omega, +, -, 0)$ and $(A, +, -, 0, \mathbb{Z}[\Sigma])$ are term equivalent. Now for $m \in \mathcal{M}_v$ let $t(v, x_1, \dots, x_n) \in \mathcal{M}$ be such that $\bar{a}(t, v) = m$. For $b \in A$ we have $mb = t(b, o, \dots, o)$. This yields that the operation corresponding to m is bijective on A . Thus (A, Ω) is a reduct of $(M, +, -, 0, R^{\mathcal{M}}) = (A, +, -, 0, R^{\mathcal{M}})$.

Case 3: $\Omega_{>1} \neq \emptyset$ and $\Omega_0 = \emptyset$.

Here the results from Section 4 will be applied. The notation introduced there will be used as well. Because \mathcal{M} satisfies (Ax) and (A, Ω) is an \mathcal{M} -polyquasigroup, for $\omega \in \Omega_{>1}$ we have $\omega(A, \dots, A) = A$. Thus, by Theorem 11 and 28 (2), the mapping $a \mapsto a/\Theta$ is an embedding from A into $N(A)/\Theta$ (see Remark 22).

Define a binary relation $\Theta^{\mathcal{M}}$ on the set $N(A)$ by

$$p \Theta^{\mathcal{M}} q \iff \left\{ (\exists m \in \mathcal{M}_v, r \in N(A)) \quad mp + r \Theta mq + r \right\}.$$

This relation is a congruence of the semimodule $(N(A), +, 0, \mathbb{N}[\Sigma])$. Indeed, if

$$mp + r \Theta mq + r \quad \text{and} \quad m'p' + r' \Theta m'q' + r',$$

then

$$\begin{aligned} mm'(p + p') + mr' + m'r &= m'(mp + r) + m(m'p' + r') \\ \Theta m'(mq + r) + m(m'q' + r') &= mm'(q + q') + mr' + m'r. \end{aligned}$$

Moreover, if $k \in \mathbb{N}[\Sigma]$, then

$$mkp + kr = k(mp + r) \Theta k(mq + r) = mkq + kr.$$

We claim that $\Theta^{\mathcal{M}} \cap A^2 = \text{id}_A$. So consider $a, b \in A$ such that $a \Theta^{\mathcal{M}} b$. This means that for some $m \in \mathcal{M}_v$ and $r \in N(A)$ we have

$$ma + r \Theta mb + r.$$

Assume that $m = \bar{a}(t_0, v)$, where $t_0(v, x_1^0, \dots, x_{i_0}^0) \in \mathcal{M}$, and $r = \sum_{k=1}^l \bar{\gamma}_k c_k$, where $\gamma_k \in \Sigma^*$ and $c_k \in A$. For $1 \leq k \leq l$ let $t_k(x_0^k, \dots, x_{i_k}^k)$ be a term such that $\bar{a}(t_k, x_0^k) = \bar{\gamma}_k$. Because \mathcal{M} is proper, there exist a term $\eta(x_1, x_2, x_3)$ and

distinct variables $y, z \neq v$ such that $\eta(v, y, z), \eta(y, v, z) \in \mathcal{M}$. Let $\delta = a(\eta, x_1)$ and $\epsilon = a(\eta, x_2)$. Let $s(x_0, \dots, x_l, z)$, be a term such that $a(s, x_k) = \epsilon^k \delta \epsilon^{l-k}$ and $s(x_1, \dots, x_{k-1}, v, x_{k+1}, x_l, z) \in \mathcal{M}$ for all $k \leq l$. Such term may be constructed by composing η with itself l times, and then by putting variables x_k under the address $\epsilon^k \delta \epsilon^{l-k}$ for $k \leq l$, and the variable z under all other addresses. Define

$$t(v, x_0^1, x_0^2, \dots, x_0^l, x_1^0, \dots, x_{i_0}^0, x_1^1, \dots, x_l^l, z) = s(t_0, t_1, \dots, t_l, z) \in \mathcal{M}.$$

Let d be any element of A . Recall that the symbol t_N denotes the term operation determined by the term t in $(N(A), \Omega)$. We have

$$\begin{aligned} t_N(a, c_1, \dots, c_l, d, \dots, d) &= \bar{\delta} \bar{\epsilon}^l m a + \bar{\epsilon} \bar{\delta} \bar{\epsilon}^{l-1} \bar{\gamma}_1 c_1 + \dots + \bar{\epsilon}^l \bar{\delta} \bar{\gamma}_l c_l + p \\ &= \bar{\delta} \bar{\epsilon}^l (m a + r) + p \\ &= \Theta \bar{\delta} \bar{\epsilon}^l (m b + r) + p \\ &= \bar{\delta} \bar{\epsilon}^l m b + \bar{\epsilon} \bar{\delta} \bar{\epsilon}^{l-1} \bar{\gamma}_1 c_1 + \dots + \bar{\epsilon}^l \bar{\delta} \bar{\gamma}_l c_l + p \\ &= t_N(b, c_1, \dots, c_l, d, \dots, d). \end{aligned}$$

Here $p = t_N(0, \dots, 0, d, \dots, d)$, where 0 appears in the first $l + 1$ slots. This and Lemma 19 yields that

$$t(a, c_1, \dots, c_l, d, \dots, d) = t(b, c_1, \dots, c_l, d, \dots, d) \quad \text{in } (A, \Omega).$$

Finally, by \mathcal{M} -cancellativity of (A, Ω) , we obtain $a = b$. This proves the claim.

Let $(G(X), +, 0, \mathbb{N}[\Sigma])$ be the absolutely free algebra over the set X of the type of semimodules over $(\mathbb{N}[\Sigma], +, 0, \cdot, 1)$. Let $\mathcal{N} \subseteq G(X)$ be the monoid of terms generated by terms $mv + x$, where $m \in \mathcal{M}_v$. The semimodule $(N(A), +, 0, \mathbb{N}[\Sigma]) / \Theta^{\mathcal{M}}$ is strongly entropic and \mathcal{N} -cancellative. Thus we can use Theorem 28 again to embed it into a semimodule $(M, +, 0, \mathbb{N}[\Sigma])$. This semimodule is an \mathcal{N} -polyquasigroup, and hence has a structure of the module $(M, +, -, 0, R^{\mathcal{M}})$. \square

7. CATEGORICAL WRAPPING

The construction presented in the previous section provides a left adjoint functor.

Let \mathbf{V} and \mathbf{W} be two varieties of types $\tau: \Omega \rightarrow \mathbb{N}$ and $\delta: \Phi \rightarrow \mathbb{N}$ respectively. By a *tensor product* of varieties \mathbf{V} and \mathbf{W} we mean the variety $\mathbf{V} \otimes \mathbf{W}$ of type $\tau \dot{\cup} \delta: \Omega \dot{\cup} \Phi \rightarrow \mathbb{N}$ satisfying identities true in \mathbf{V} , identities true in \mathbf{W} and all $\varepsilon_{\mu, \nu}$, where $\mu \in \Omega$ and $\nu \in \Phi$ [6].

Let \mathbf{CMon} be the variety of commutative monoids and \mathbf{SMod}_S be the variety of semimodules over a semiring $(S, +, 0, \cdot, 1)$. We have the following term equivalences of varieties

$$\mathbf{E} \otimes \mathbf{CMon} \simeq \mathbf{SMod}_{\mathbb{N}[\Sigma]} \simeq \mathbf{SE} \otimes \mathbf{CMon}.$$

Let $U: \mathbf{E} \otimes \mathbf{CMon} \rightarrow \mathbf{E}$ be the forgetful functor and H be its left adjoint. For a monoid \mathcal{M} of terms consider the forgetful functor $V^{\mathcal{M}}: \mathbf{Mod}_{R^{\mathcal{M}}} \rightarrow \mathbf{SMod}_{\mathbb{N}[\Sigma]}$, with the variety of modules over $(R^{\mathcal{M}}, +, -, 0, \cdot, 1)$ as a domain, and its left adjoint $K^{\mathcal{M}}$. Now consider the following composition of adjunctions

$$\begin{array}{c}
\begin{array}{ccc}
& & M^{\mathcal{M}} \\
& \nearrow & \\
& & \longrightarrow \\
\mathbf{E} & \xleftrightarrow{H} & \mathbf{E} \otimes \mathbf{CMon} \simeq \mathbf{SMod}_{\mathbb{N}_C[\Sigma]} \xleftrightarrow{K^{\mathcal{M}}} \mathbf{Mod}_{R^{\mathcal{M}}} \\
& \xleftarrow{U} & \\
& & \longleftarrow \\
& & V^{\mathcal{M}} \\
& \searrow & \\
& & W^{\mathcal{M}}
\end{array}
\end{array}$$

Theorem 30. *Let \mathcal{M} be a monoid of terms. Then the functor $M^{\mathcal{M}}$ is a left adjoint to the forgetful functor $W^{\mathcal{M}}$. Moreover, if \mathcal{M} is proper and satisfies (Ax), then for each \mathcal{M} -cancellative entropic algebra (A, Ω) the unit of this adjunction*

$$\eta_{(A, \Omega)}: (A, \Omega) \rightarrow W^{\mathcal{M}}(M^{\mathcal{M}}(A, \Omega))$$

is injective.

A similar situation appeared in Section 4. The assignment $(A, \Omega) \mapsto (N, +, 0, T)$ described there may be considered as an object part of the functor from \mathbf{E} into \mathbf{SMod}_T which is a left adjoint to an appropriate forgetful functor.

8. DISCUSSION

In conclusion, we would like to discuss some problems which appeared during the work on this article.

Recall that an algebra is *quasi-affine* if it is a subreduct of an algebra polynomially equivalent to a module [19]. It may be proved that if entropic algebra is quasi-affine, then it is quasi-affine over a commutative ring. Moreover, each quasi-affine algebra without constants is a subreduct of a module [34]. These facts suggest that the following question may have a positive answer.

Problem 31. Is it true that each quasi-affine entropic algebra (without constants) is a subreduct of a module over a commutative ring?

Let \mathcal{M} be a proper monoid of terms. Each \mathcal{M} -cancellative entropic algebra is quasi-affine [33, Theorem 8.6]. Here is a subproblem.

Problem 32. Let \mathcal{M} be a proper monoid of terms. Is it true that every entropic \mathcal{M} -cancellative algebra is a subreduct of a module over a commutative ring?

Note that in the proof of Theorem 29 we used the fact that \mathcal{M} satisfies condition (Ax) only once, when we showed that the considered algebra is a subreduct of a semimodule over a commutative semiring. Hence to show that Problem 32 has a positive solution it would be enough to prove that each quasi-affine algebra over a commutative ring without constants is a subreduct of a semimodule over a commutative semiring. Note however that Remark 23 suggests that Problems 31 and 32 may have negative solutions.

A characterization of subreducts of (semi)modules over arbitrary (semi)rings is a much easier issue. In fact each algebra without constants is a subreduct of a semimodule [10]. As mentioned before, each quasi-affine algebra without constants is a subreduct of a module [34].

The last problem concerns subreducts of vector spaces.

Problem 33. Characterize subreducts of vector spaces.

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FACULTY OF MATHEMATICS AND INFORMATION SCIENCES, WARSAW UNIVERSITY OF TECHNOLOGY, WARSAW, POLAND

EDUARD ČECH CENTER, CHARLES UNIVERSITY, PRAGUE, CZECH REPUBLIC
E-mail address: `m.stronkowski@mini.pw.edu.pl`